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The $4d$ Superconformal Index from q-deformed $2d$ Yang-Mills

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We identify the $2d$ topological theory underlying the $\mathcal{N} = 2$ $4d$ superconformal index with an explicit model: q-deformed $2d$ Yang-Mills. By this route we are able to evaluate the index of some strongly-coupled $4d$ SCFTs, such as Gaiotto's T_N theories.

INTRODUCTION

In this letter we describe a new powerful duality, relating physics in four and in two dimensions. We will argue that for a large class of *four-dimensional superconformal* gauge theories, non-trivial information about the operator spectrum is captured by correlators of a *two-dimensional non-supersymmetric* gauge theory. The $4d$ side of the duality is generically strongly-coupled, and difficult to analyze directly; on the other hand calculations on the $2d$ side will be explicit and algorithmic. Thus our conjecture gives new information about strongly-coupled $4d$ field theories.

Our proposal is in the same spirit as the Alday-Gaiotto-Tachikawa (AGT) relation between the partition function of a $4d$ $\mathcal{N} = 2$ gauge theory on S^4 and a correlator in $2d$ Liouville/Toda theory [1, 2]. In our case, the $4d$ observable is a (twisted) supersymmetric partition function of an $\mathcal{N} = 2$ superconformal field theory on $S^3 \times S^1$, also known as the superconformal index. We will focus on a "reduced" index that depends on a single fugacity q. On the $2d$ side, instead of Liouville/Toda we have the *zero-area limit* of q-deformed Yang-Mills theory. The topological nature of this $2d$ theory dovetails with the independence of the $4d$ index on the gauge theory moduli.

We begin by reviewing the $4d$ side of the duality. The full $\mathcal{N} = 2$ superconformal index is defined as [3]

$$\mathcal{I} = \mathrm{Tr}(-1)^F p^{\frac{E-R}{2}+j_1} q^{\frac{E-R}{2}-j_1} u^{-(r+R)}\,, \qquad (1)$$

where the trace is over the states of the theory on S^3 (in the usual radial quantization) and F the fermion number. The symbol E stands for the conformal dimension, (j_1, j_2) for the Cartan generators of the $SU(2)_1 \otimes SU(2)_2$ isometry group, and $(R\,, r)$ for the Cartan generators of the $SU(2)_R \otimes U(1)_r$ R-symmetry. The fugacities p, q, and u keep track of the maximal set of quantum numbers commuting with a single real supercharge, $\mathcal{Q} \equiv \tilde{\mathcal{Q}}_{1\dot{-}}$, which with no loss of generality has been chosen to have $R = \frac{1}{2}$, $r = -\frac{1}{2}$, $j_1 = 0$, $j_2 = -\frac{1}{2}$ and (of course) $E = \frac{1}{2}$. Only states that obey $2\{\mathcal{Q}, \mathcal{Q}^\dagger\} = E - 2j_2 - 2R + r = 0$ contribute to the index. Note that the variables p, q, and u are related to t, y, v of [4] as $p = t^3 y$, $q = \frac{t^3}{y}$ and $u = \frac{v}{t}$.

For a theory with a weakly-coupled Lagrangian description the index is computed explicitly by a matrix integral,

$$\mathcal{I}(p, q, u; V) = \int [dU] \qquad (2)$$
$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \sum_j f^{(j)}(p^n, q^n, u^n)\, \chi_{\mathcal{R}_j}(U^n,\, V^n)\right).$$

Here U denotes an element of the gauge group, with $[dU]$ the invariant Haar measure, and V an element of the flavor group. The sum is over the different $\mathcal{N} = 2$ supermultiplets appearing in the Lagrangian, with $\mathcal{R}_j$ the representation of the j-th multiplet under the flavor and gauge groups and $\chi_{\mathcal{R}_j}$ the corresponding character. The functions $f^{(j)}$ are the "single-letter" partition functions, $f^{(j)} = f^{vect}$ or $f^{(j)} = f^{chi}$ according to whether the j-th multiplet is an $\mathcal{N} = 2$ vector or $\mathcal{N} = 2$ $\frac{1}{2}$-hypermultiplet. They are easily evaluated [3]:

$$f^{vect}(p, q, u) = \frac{(u - \frac{1}{u})\sqrt{pq} - (p + q) + 2pq}{(1 - p)(1 - q)}\,, \qquad (3)$$

$$f^{chi}(p, q, u) = \frac{(pq)^{\frac{1}{4}} \frac{1}{\sqrt{u}} - (pq)^{\frac{3}{4}} \sqrt{u}}{(1 - p)(1 - q)}\,. \qquad (4)$$

We will focus on a *reduced* index, by setting

$$u = 1, \qquad p = q\,, \qquad (5)$$

which leads to the significant simplification

$$f^{vect} = \frac{-2q}{1 - q}, \qquad f^{chi} = \frac{q^{\frac{1}{2}}}{1 - q}\,. \qquad (6)$$

We consider a class of $\mathcal{N} = 2$ $4d$ superconformal theories (SCFTs) constructed from a set of elementary building blocks [5]. The building blocks are isolated SCFTs with flavor symmetry $G_1 \otimes G_2 \otimes G_3$, $G_i \subseteq SU(N)$ for given N. In the simplest case of $N = 2$, the only building block is the free $\frac{1}{2}$-hypermultiplet in the tri-fundamental representation of the $SU(2)^3$ flavor group. For $N > 2$ most of the building blocks are intrinsically strongly-interacting theories with no Lagrangian description. One can "glue together" two building blocks by gauging a

common $SU(N)$ flavor symmetry. Iterating this procedure one constructs a large class of $\mathcal{N}=2$ gauge theories, the $SU(N)$ "generalized quivers" [5]. There is a geometric interpretation of this construction, where one regards the building blocks as three-punctured spheres, with the punctures associated to the flavor symmetries; the gluing operation is performed by connecting the punctures with cylinders. The complex structure moduli of the resulting punctured Riemann surface correspond to the complexified gauge couplings. The same punctured Riemann surface can often be obtained by following several different gluing paths (different pairs-of-pants decompositions). The generalized quiver theories associated to different decompositions of the same surface are related by S-dualities [5].

The index of a generalized quiver can be written in terms of the index of its constituents. We parametrize the index of an elementary building block (3-punctured sphere) by "structure constants" $\mathcal{I}_N(\mathbf{x_1},\mathbf{x_2},\mathbf{x_3})$ where $\mathbf{x}_i$ are fugacities dual to the Cartan subgroup of G_i: except in special cases these are *a priori* unknown functions. On the other hand we can easily write the index $\eta_N(\mathbf{x})$ of the $SU(N)$ vector multiplets used in the gluing (propagators),

$$\eta_N(\mathbf{x})=\exp\left[-2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\chi_{adj}(\mathbf{x}^n)\right].$$

For example, gluing two 3-punctured spheres with one cylinder one obtains the following index

$$\int[dU(\mathbf{x})]\,\mathcal{I}_N(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x})\,\eta_N(\mathbf{x})\,\mathcal{I}_N(\mathbf{x},\mathbf{x}_3,\mathbf{x}_4)\,. \tag{7}$$

By defining a metric

$$\eta_N(\mathbf{x}_1,\mathbf{x}_2)\equiv\eta_N(\mathbf{x}_1)\sum_{\mathcal{R}}\chi_{\mathcal{R}}(\mathbf{x}_1)\,\chi_{\mathcal{R}}(\mathbf{x}_2)\,, \tag{8}$$

with $\mathcal{R}$ running over irreducible and finite representations of $SU(N)$, we can re-write (7) as

$$\mathcal{I}_N(\mathbf{x_1},\mathbf{x_2},\mathbf{x})\cdot\,\eta_N(\mathbf{x},\mathbf{x}')\cdot\,\mathcal{I}_N(\mathbf{x}',\mathbf{x_3},\mathbf{x_4})\,, \tag{9}$$

where $\cdot$ multiplication means integration over the Haar measure. S-duality then implies that the metric and structure constants form an associative algebra and thus a $2d$ topological field theory (TQFT) [4]. (Strictly speaking, the state-space at each puncture, which is spanned by G_i representations, is infinite-dimensional, so one must slightly relax the standard mathematical axioms for a TQFT.) Associativity was directly verified for the $SU(2)$ and $SU(3)$ generalized quiver theories in [4, 6], for generic values of the fugacities p, q and u. In the following we will identify the $2d$ topological theory implicitly defined by the reduced index with an explicit model: q-deformed Yang-Mills (qYM) in the zero-area limit.

$SU(2)$ GENERALIZED QUIVERS

Let us start with the simplest case, the $SU(2)$ quivers. Here the building blocks are free tri-fundamental $\frac{1}{2}$-hypermultiplets,

$$\mathcal{I}_{222}(a_1,a_2,a_3)=\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^{\frac{1}{2}n}}{1-q^n}\chi_{\Box}(a_1^n)\chi_{\Box}(a_2^n)\chi_{\Box}(a_3^n)\right].$$

Remarkably, one can prove (*e.g.* by comparing analytic properties) that $\mathcal{I}_{222}(a_1,a_2,a_3)$ admits the equivalent representation

$$\mathcal{I}_{222}(a_1,a_2,a_3)= \tag{10}$$
$$\frac{(q;q)_\infty}{1-q}\prod_{i=1}^{3}\eta_2^{-\frac{1}{2}}(a_i)\sum_{\mathcal{R}}\frac{\chi_{\mathcal{R}}(a_1)\,\chi_{\mathcal{R}}(a_2)\,\chi_{\mathcal{R}}(a_3)}{[|\mathcal{R}|]_q}\,.$$

Here $(q;q)_\infty\equiv\prod_{i=1}^{\infty}(1-q^i)$. The sum is over irreducible $SU(2)$ representations $\mathcal{R}$, with $|\mathcal{R}|$ denoting the dimension of the representation. The $SU(2)$ characters are

$$\chi_{\mathcal{R}}(a)=\frac{a^{|\mathcal{R}|}-a^{-|\mathcal{R}|}}{a-a^{-1}}\,. \tag{11}$$

Finally the symbol $[x]_q$ denotes the *q-deformed* number,

$$[x]_q\equiv\frac{q^{-\frac{x}{2}}-q^{\frac{x}{2}}}{q^{-\frac{1}{2}}-q^{\frac{1}{2}}}\,. \tag{12}$$

The structure constants contain the factors $\prod_i\eta_2^{-1/2}(a_i)$, which cancel with the metric $\eta_2(a_i)$ when two punctures are glued. It is then natural to define rescaled structure constants and metric,

$$\hat{\mathcal{I}}_{222}(a_1,a_2,a_3)=\mathcal{N}_{222}(q)\sum_{\mathcal{R}}\frac{\chi_{\mathcal{R}}(a_1)\,\chi_{\mathcal{R}}(a_2)\,\chi_{\mathcal{R}}(a_3)}{[|\mathcal{R}|]_q}\,,$$
$$\hat{\eta}_2(a,\,b)=\sum_{\mathcal{R}}\chi_{\mathcal{R}}(a)\,\chi_{\mathcal{R}}(b)\,, \tag{13}$$

where $\mathcal{N}_{222}(q)=(q;q)_\infty/(1-q)$. Up to the overall normalization $\mathcal{N}_{222}$, these are precisely the structure constants and metric of $2d$ qYM in the zero area limit [7, 8]!

Note that $[n]_q=\chi_n(q^{1/2})$. This implies that by setting one of the $SU(2)$ fugacities to $q^{1/2}$ we "close" a puncture,

$$\hat{\mathcal{I}}_{222}(a,b,q^{1/2})=\mathcal{N}_{222}(q)\;\;\hat{\eta}_2(a,\,b)\,.$$

Applying this procedure again, we close another puncture and obtain the one-punctured sphere (the cap). For higher-rank groups we will encounter a similar procedure: setting some combination of the flavor fugacities to $q^{1/2}$ one obtains punctures with reduced flavor symmetry.

$SU(3)$ GENERALIZED QUIVERS

Next let us consider the $SU(3)$ generalized quivers. Here two new generic features appear. First, the basic building block is an interacting theory with no Lagrangian description, the E_6 SCFT [5, 9]. Second, there

is more than one type of puncture: in addition to the *maximal* $SU(3)$ flavor puncture there is a puncture with reduced flavor symmetry, $U(1)$ [5].

The representations of $SU(N)$ are parametrized by N integers $\lambda_1 \geq \lambda_2 ... \geq \lambda_{N-1} \geq \lambda_N = 0$, the row lengths of the corresponding Young diagram. The q-deformed dimension of the representation is

$$\dim_q \mathcal{R}_{\underline{\lambda}} = \prod_{i<j} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}\,, \tag{14}$$

and the characters are given by Schur polynomials

$$\chi_{\underline{\lambda}}(\mathbf{x}) = \frac{\det\left({x_i}^{\lambda_j + k - j}\right)}{\det\left({x_i}^{k-j}\right)}\,. \tag{15}$$

Specializing to $SU(3)$ we can parametrize all the Young diagrams by (λ_1, λ_2). We observe again that the q-dimension of a representation is equal to the group character with a particular choice of fugacities,

$$\chi_{\lambda_1,\lambda_2}(q, 1, q^{-1}) = \dim_q \mathcal{R}_{\lambda_1,\lambda_2}\,. \tag{16}$$

Three Maximal Punctures

The sphere with three maximal punctures corresponds to the strongly coupled E_6 SCFT (the $SU(3)^3$ flavor symmetry is accidentally enhanced to E_6.) This theory has no Lagrangian description and thus we do not have a direct way to compute its index. However, this index was computed [6] indirectly by employing Argyres-Seiberg duality [9]. Inspired by the $SU(2)$ case, we conjecture that the index $\mathcal{I}_{E_6}(\{\mathbf{x}_i\}_{i=1}^3)$ of the E_6 SCFT is proportional to the structure constants $C_{SU(3)_q}$ of q-deformed $SU(3)$ Yang-Mills,

$$\mathcal{I}_{E_6}(\mathbf{x_i}) = \left[\prod_{i=1}^{3} \eta^{-\frac{1}{2}}(\mathbf{x}_i)\right] \mathcal{N}_{333}(q)\, C_{SU(3)_q}(\mathbf{x_i})\,,$$

where

$$C_{SU(3)_q}(\mathbf{x_i}) = \sum_{0 \leq \lambda_2 \leq \lambda_1}^{\infty} \frac{\chi_{\lambda_1,\lambda_2}(\mathbf{x}_1)\chi_{\lambda_1,\lambda_2}(\mathbf{x}_2)\chi_{\lambda_1,\lambda_2}(\mathbf{x}_3)}{\dim_q \mathcal{R}_{\lambda_1,\lambda_2}},$$

and $\mathcal{N}_{333}(q)$ a normalization factor. Using *Mathematica*, we have checked this proposal against the results of [6] to several orders in q, and in the process determined the normalization to be

$$\mathcal{N}_{333}(q) = \frac{(q;q)_\infty^2}{(1-q)^2(1-q^2)}\,. \tag{17}$$

Two Maximal and One $U(1)$ Puncture

Another building block is given by a sphere with two $SU(3)$ punctures and one $U(1)$ puncture. This corresponds to a free hypermultiplet in the bi-fundamental of $SU(3)^2$ and charged under the $U(1)$. The index of this theory is explicitly given by

$$\mathcal{I}_{331}(\mathbf{x_1}, \mathbf{x_2};\ a) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{\frac{1}{2}n}}{1-q^n} \chi_{hyp}({\mathbf{x_1}}^n, {\mathbf{x_2}}^n;\ a^n)\right],$$

where the flavor character is

$$\chi_{hyp}(\mathbf{x_1}, \mathbf{x_2};\ a) = \sum_{i,j} (x_1^i x_2^j a + \frac{1}{x_1^i x_2^j a})\,. \tag{18}$$

One can verify by series expansion in q that

$$\begin{aligned} \mathcal{I}_{331}(\mathbf{x_1}, \mathbf{x_2};\ a) = C_{SU(3)_q}(\mathbf{x_1}, \mathbf{x_2};\ a) \times \\ \frac{\prod_{i=1}^2 \eta^{-\frac{1}{2}}(\mathbf{x}_i)}{\prod_{\ell=1}^2 (1-q^\ell)} \exp\left[\sum_{n=1}^{\infty} \frac{q^{\frac{3}{2}n}}{1-q^n} \frac{a^{3n}+a^{-3n}}{n}\right], \end{aligned} \tag{19}$$

with

$$\begin{aligned} C_{SU(3)_q}(\mathbf{x_1}, \mathbf{x_2};\ a) = \\ \sum_{0 \leq \lambda_2 \leq \lambda_1}^{\infty} \frac{\chi_{\lambda_1,\lambda_2}(\mathbf{x_1})\, \chi_{\lambda_1,\lambda_2}(\mathbf{x_2})\, \chi_{\lambda_1,\lambda_2}(a\, q^{1/2}, a q^{-1/2}, a^{-2})}{\dim_q \mathcal{R}_{\lambda_1,\lambda_2}}\,. \end{aligned} \tag{20}$$

Note that this result can be recovered by starting from the structure constant with maximal punctures and "partially closing" one of the punctures by embedding $SU(2)$ fugacities $(q^{\frac{1}{2}}, q^{-\frac{1}{2}})$ into fugacities of $SU(3)$.

GENERAL STATEMENT

The generic building block of a higher-rank quiver is an interacting SCFT with no Lagrangian description. Unlike the case of $SU(2)$ and $SU(3)$ quivers it is very hard to calculate the index of these theories, either directly or indirectly. However, we can naturally extrapolate the relation to $2d$ qYM to higher-rank groups.

We conjecture that the reduced index of the theory corresponding to sphere with three maximal punctures (the T_N theory of [5]) is

$$\mathcal{I}_{T_N}(\mathbf{x_i}) = \frac{(q;q)_\infty^{N-1} \prod_{i=1}^3 \eta^{-\frac{1}{2}}(\mathbf{x}_i)}{\prod_{\ell=1}^{N-1}(1-q^\ell)^{N-\ell}}\, C_{SU(N)_q}(\mathbf{x_i})$$

where

$$C_{SU(N)_q}(\mathbf{x_i}) = \sum_{\mathcal{R}} \frac{1}{\dim_q \mathcal{R}}\, \chi_{\mathcal{R}}(\mathbf{x}_1)\, \chi_{\mathcal{R}}(\mathbf{x}_2)\, \chi_{\mathcal{R}}(\mathbf{x}_3)$$

are the structure constant of $SU(N)$ qYM. The sum is over irreducible $SU(N)$ representations and $\{\mathbf{x}_i\}$ are the fugacities dual to the Cartan subgroup.

This conjecture can be tested against the numerous S-dualities of the generalized quivers [5]. For instance, a linear superconformal quiver theory with two $SU(4)$ nodes admits a dual description in terms of T_4 coupled to $SU(3)$ gauge theory which in turn is coupled to an $SU(2)$ gauge theory with a single hypermultiplet. We

have checked, in the q expansion, that the indices on both sides of the duality indeed match if one uses our conjecture for the T_4 index.

Another test is to compare with physical expectations for the spectrum of protected operators. A class of protected operators in the T_N theories are the Higgs branch operators [10]. These come in two families: $E = 2, R = 1$ in flavor representation $(adj, 1, 1) \oplus (1, adj, 1) \oplus (1, 1, adj)$ and $E = N-1, R = \frac{N-1}{2}$ in representation $(N, N, N) \oplus (\bar{N}, \bar{N}, \bar{N})$. It is straightforward to see that these operators appear in our conjecture for the index: the first family comes from the $\eta(\mathbf{x})^{-\frac{1}{2}}$ factors, and the second from the $\chi_{\Box}(\mathbf{x}_1)\chi_{\Box}(\mathbf{x}_2)\chi_{\Box}(\mathbf{x}_3)$ and $\chi_{\overline{\Box}}(\mathbf{x}_1)\chi_{\overline{\Box}}(\mathbf{x}_2)\chi_{\overline{\Box}}(\mathbf{x}_3)$ terms in $C_{SU(N)_q}$.

We can generalize the conjecture to the structure constants with two maximal punctures and one $U(1)$ puncture,

$$\mathcal{I}_{NN1}(\mathbf{x_1}, \mathbf{x_2}, a) = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{\frac{1}{2}n}}{1-q^n} \chi_{hyp}(\mathbf{x_1}^n, \mathbf{x_2}^n;\ a^n)\right] = \frac{C_{SU(N)_q}(\mathbf{x_1}, \mathbf{x_2};\ a)}{\prod_{i=1}^{2} \eta^{\frac{1}{2}}(\mathbf{x}_i) \prod_{\ell=1}^{N-1}(1-q^\ell)} \exp\left[\sum_{n=1}^{\infty} \frac{q^{\frac{N}{2}n}}{1-q^n} \frac{a^{Nn} + a^{-Nn}}{n}\right],$$

where structure constants $C_{SU(N)_q}(\mathbf{x_1}, \mathbf{x_2}; a)$ are

$$C_{SU(N)_q}(\mathbf{x_1}, \mathbf{x_2}; a) = \sum_{\mathcal{R}} \frac{1}{\dim_q \mathcal{R}} \chi_{\mathcal{R}}(\mathbf{x}_1)\, \chi_{\mathcal{R}}(\mathbf{x}_2)\, \chi_{\mathcal{R}}(a q^{\frac{N-2}{2}}, .., a q^{-\frac{N-2}{2}}, a^{1-N}) . \tag{21}$$

Again we have verified this conjecture in the q-expansion.

Generic punctures are classified [5] by the embeddings

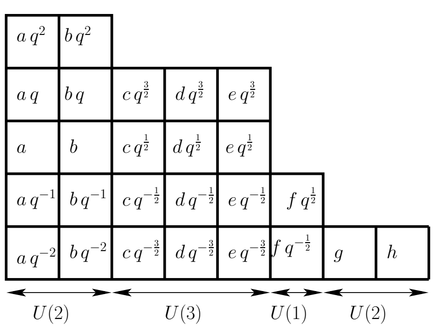

FIG. 1. An example of the rule to associate flavor fugacities for a non-maximal puncture. Illustrated here is a puncture for $N = 26$ with flavor symmetry $S(U(3)U(2)^2U(1))$. The $S(\dots)$ constraint imposes $(ab)^5(cde)^4 f^2 gh = 1$.

$SU(2) \subset SU(N)$, which are specified by the decomposition of the fundamental of $SU(N)$ into $SU(2)$ representation. (In the terminology of [11], we focus on regular punctures). This information can be encoded into a Young diagram with N boxes, where the height of each column denotes the dimension of an $SU(2)$ representation. The commutant of this embedding is the flavor symmetry associated to the puncture. The maximal puncture corresponds to a single-row diagram, the closed puncture (*i.e.* no puncture) corresponds to a single-column diagram, and the $U(1)$ puncture to a two-column diagram with $N-1$ boxes in the first column and a single box in the second column. The Young diagram in Fig. 1 exemplifies a non-maximal puncture for $N = 26$ with $S(U(3)U(2)^2U(1))$ flavor symmetry. We are lead to the following conjecture for the index of a theory with three generic punctures corresponding to Young diagarms λ_i

$$\mathcal{I}(\Lambda_1, \Lambda_2, \Lambda_3) = \mathcal{N}_{\lambda_1,\lambda_2,\lambda_3}(q) \prod_{i=1}^{3} \mathcal{A}_{\lambda_i}(\Lambda_i) \times \sum_{\mathcal{R}} \frac{1}{\dim_q \mathcal{R}}\, \chi_{\mathcal{R}}(\Lambda_1)\, \chi_{\mathcal{R}}(\Lambda_2)\, \chi_{\mathcal{R}}(\Lambda_3) ,$$

with Λ_i labeling an association of flavor fugacities according to the Young diagram λ_i. The rule to associate the flavor fugacities to the $SU(N)$ fugacities is illustrated in Fig. 1. For all maximal punctures we have given the normalization factors ($\mathcal{N}$ and $\mathcal{A}$) above, while for generic punctures these factors can be in principle obtained by employing different S-dualities of the quivers [5]. As an example, consider the E_7 SCFT which is given by a sphere with two maximal punctures of $SU(4)$ and one square Young diagram with four boxes. Following the above procedure and fixing the normalization from the relevant Argyres-Seiberg duality [9], we are led to propose

$$\mathcal{I}_{E_7}(\mathbf{x}, \mathbf{y}; a) = \frac{\exp\left[\sum_{n=1}^{\infty} \frac{q^n(1+q^n)}{1-q^n} \frac{a^{2n}+a^{-2n}}{n}\right]}{\eta^{\frac{1}{2}}(\mathbf{x})\eta^{\frac{1}{2}}(\mathbf{y})(1-q)(1-q^2)^2(1-q^3)} \times \sum_{\mathcal{R}} \frac{\chi_{\mathcal{R}}(\mathbf{x})\, \chi_{\mathcal{R}}(\mathbf{y})\, \chi_{\mathcal{R}}(q^{\frac{1}{2}}a, q^{-\frac{1}{2}}a, q^{\frac{1}{2}}/a, q^{-\frac{1}{2}}/a)}{\dim_q \mathcal{R}} ,$$

Here $\mathbf{x}$, $\mathbf{y}$ label the two sets of $SU(4)$ fugacities and a the $SU(2)$ fugacity. The summation, as usual, is over finite irreducible representations of $SU(4)$. We have verified perturbatively in q that this expression is indeed E_7 covariant – a tight check of our logic.

DISCUSSION

We have given compelling evidence that the reduced superconformal index of an $\mathcal{N} = 2$ generalized $SU(N)$ quiver theory is exactly computed by a correlator in $2d$ $SU(N)_q$ Yang-Mills. This duality is new tool to investigate interacting field theories without a Lagrangian description. For example, it should be useful to study the constraints obeyed by the Higgs branch operators, generalizing to $N > 3$ the analysis of [12]. Two-dimensional qYM first appeared in a physical setting in the context of counting BPS states [7], and it would be interesting to find a relation with our work. An obvious question is whether our results can be generalized to the full index,

with all fugacities turned on. It is already remarkable that the *known* structure constants of the $SU(2)$ quivers implicitly define a (q, p, u) deformation of $SU(2)$ Yang-Mills. Work is in progress in investigating the nature of this deformation, in order to extrapolate it to $N > 2$. The q and p fugacities appear on a symmetric footing, in a way which is strongly suggestive of an elliptic, or "dynamical", deformation of the quantum group structure $SU(N)_q$ that we have uncovered for $p = q$, $u = 1$. Indeed the full index is most elegantly expressed [13] in terms of elliptic Gamma functions [14]. Finally, a more conceptual understanding of the duality would be very desirable. As for the AGT correspondence [1], the existence, but not the details, of a $4d/2d$ relation can be traced to the definition of the $4d$ SCFT as the infrared limit of the $6d$ (2,0) theory on a Riemann surface. Whether this intuition can be turned into a microscopic derivation remains to be seen.

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